QUESTIONS FOR PRACTICE SOLUTIONS

Chapter-1 Integration and Its Applications

1. Evaluate: $\int \frac{e^x(1+x)}{(1+xe^x)^2}dx$

Sol. We have,

$$I = \int \frac{e^x(1+x)}{(1+xe^x)^2}dx$$

Let $1+xe^x = t \Rightarrow (xe^x + e^x)dx = dt$

$$\Rightarrow e^x(1+x)dx = dt$$

$$I = \int \frac{dt}{t^2} = \int t^{-2}dt = \frac{t^{-2+1}}{-2+1} + C$$

$$\therefore \quad = -\frac{1}{t} + C = \frac{-1}{1+xe^x} + C$$

2. Evaluate: $\int \frac{2x}{\sqrt[3]{x^2+1}}dx$

Sol. We have, $I = \int \frac{2x}{\sqrt[3]{x^2+1}}dx$

Let $x^2+1 = t \Rightarrow 2xdx = dt$

$$\therefore \quad I\int \frac{dt}{\sqrt[3]{t}} = \int \frac{dt}{t^{1/3}} = \int t^{-\frac{1}{3}}\,dt$$

$$= \frac{t^{-1/3+1}}{-\frac{1}{3}+1} + C = \frac{3}{2}t^{2/3} + C = \frac{3}{2}(x^2+1)^{2/3} + C$$

3. If $\int \left(\frac{x-1}{x^2}\right)e^x dx = f(x)e^x + C$, find the value of $f(x)$.

Sol. Given, $\int \left(\frac{x-1}{x^2}\right)e^x dx = f(x).e^x + C$

$$\Rightarrow \quad \int \left(\frac{1}{x} - \frac{1}{x^2}\right)e^x dx = f(x).e^x + C \quad \Rightarrow \quad \frac{1}{x}.e^x + C = f(x).e^x + C$$

Equating we get $f(x) = \frac{1}{x}$

[Note: $\int [f(x) + f'(x)]e^x dx = f(x)e^x + C]$

4. Evaluate : $\int \frac{(x+3)e^x}{(x+5)^3}dx$

Sol. Let $I = \int \frac{(x+3)e^x}{(x+5)^3}dx = \int \left[\frac{1}{(x+5)^2} - \frac{2}{(x+5)^3}\right]e^x dx$

It is in the form of $\int \{f(x) + f'(x)\}e^x dx$

And we know that $\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C$

$$\therefore \quad I = e^x.\frac{1}{(x+5)^2} + C$$

5. Evaluate : $\int_1^2 e^{-\log x}\, dx$

Sol. We have,

$$I = \int_1^2 e^{-\log x} dx = \int_1^2 e^{\log x^{-1}} dx$$

$$\int_1^2 x^{-1} dx = \int_1^2 \frac{1}{x} dx = [\log x]_1^2$$

$$= \log 2 - \log 1 = \log 2 - 0 = \log 2$$

6. Evaluate : $\int_{\log 2}^{\log 4} 2^x\, dx$

Sol. We have,

$$I = \int_{\log 2}^{\log 4} 2^x\, dx = \left[\frac{2^x}{\log 2}\right]_{\log 2}^{\log 4}$$

$$= \frac{2^{\log 4} - 2^{\log 2}}{\log 2}$$

7. Evaluate: $\int \frac{2^x}{\sqrt{4^x - 1}}\, dx$

Sol. We have,

$$I = \int \frac{2^x}{\sqrt{4^x - 1}} dx$$

$$\int \frac{2^x}{\sqrt{(2^x)^2 - (1)^2}} dx$$

Let $2^x = t \Rightarrow 2^x . \log 2\, dx = dt$

$$2^x dx = \frac{dt}{\log 2}$$

$$\therefore \quad I = \int \frac{dt}{\log 2 . \sqrt{t^2 - 1}} = \frac{1}{\log 2} \int \frac{dt}{\sqrt{t^2 - 1}}$$

$$= \frac{1}{\log 2} \log\left|t + \sqrt{t^2 - 1}\right| + C = \frac{\log\left|2^x + \sqrt{4^x - 1}\right|}{\log 2} + C$$

8. Evaluate: $\int \frac{dx}{x(x^3 + 8)}$

Sol. $I = \int \frac{dx}{x(x^3+8)} \quad \Rightarrow \quad I = \int \frac{x^2 dx}{x^3(x^3+8)}$

Let $x^3 = z \Rightarrow 3x^2 dx = dz$

$$\Rightarrow \quad x^2 dx = \frac{dz}{3}$$

$$\therefore \quad I = \frac{1}{3} \int \frac{dz}{z(z+8)} = \frac{1}{3 \times 8} \int \frac{(z+8) - z}{z(z+8)} dz$$

$$= \frac{1}{3 \times 8} \int \left[\frac{1}{z} - \frac{1}{z+8}\right] dz = \frac{1}{24} \int \frac{dz}{z} - \frac{1}{24} \int \frac{dz}{z+8}$$

$$= \frac{1}{24} \log|z| - \frac{1}{24} \log|z+8| + C$$

$$= \frac{1}{24} \log\left|\frac{z}{z+8}\right| + C = \frac{1}{24} \log\left|\frac{x^3}{x^3+8}\right| + C$$

9. Evaluate: $\int \frac{dx}{x(x^5+3)}$

Sol. Let $I = \int \frac{dx}{x(x^5+3)} = \int \frac{x^4 dx}{x^5(x^5+3)} = \frac{1}{5}\int \frac{5x^4 dx}{x^5(x^5+3)}$

Put $x^5 = z \Rightarrow 5x^4\, dx = dz$

$$\therefore \quad I = \frac{1}{5}\int \frac{dz}{z(z+3)} = \frac{1}{5\times 3}\int \frac{z+3-z}{z(z+3)}dz$$

$$= \frac{1}{15}\int \frac{z+3}{z(z+3)}dz - \frac{1}{15}\int \frac{z}{z(z+3)}dz$$

$$= \frac{1}{15}\int \frac{dz}{z} - \frac{1}{15}\int \frac{dz}{z+3} = \frac{1}{15}\{\log|z| - \log|z+3|\} + C$$

$$= \frac{1}{15}\log\left|\frac{z}{z+3}\right| + C = \frac{1}{15}\log\left|\frac{x^5}{x^5+3}\right| + C$$

10. Evaluate: $\int \frac{1-x^2}{x(1-2x)}dx$

Sol. We have

$$\int \frac{1-x^2}{x(1-2x)}dx = \int \frac{1-x^2}{x-2x^2}dx$$

$$= \int \frac{x^2-1}{2x^2-x}dx = \int \frac{1}{2}\left(\frac{2x^2-2}{2x^2-x}\right)dx = \frac{1}{2}\int \frac{(2x^2-x)+(x-2)}{2x^2-x}dx$$

$$= \frac{1}{2}\int\left(1+\frac{x-2}{2x^2-x}\right)dx \qquad \ldots(i)$$

By partial fraction

$$\frac{x-2}{2x^2-x} = \frac{x-2}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$$

$$x - 2 = A(2x-1) + Bx \qquad \ldots(ii)$$

Equating coefficient of x and constant term, we get

$$2A + B = 1 \quad \text{and} \quad -A = -2$$

$$\Rightarrow \quad A = 2, \; B = -3$$

$$\therefore \quad \frac{x-2}{2x^2-x} = \frac{2}{x} + \frac{3}{1-2x}$$

From equation (i)

$$\int \frac{1-x^2}{x(1-2x)}dx = \frac{1}{2}\int 1dx + \frac{1}{2}\int\left(\frac{2}{x}+\frac{3}{1-2x}\right)dx$$

$$= \frac{1}{2}x + \log|x| - \frac{3}{4}\log|1-2x| + C$$

11. Evaluate: $\int x^2 \log x\, dx$

Sol. We have,

$$I = \int \underset{\text{II}}{x^2}\underset{\text{I}}{\log x}\, dx$$

Using ILATE, we have

$$I = \log x \int x^2 dx - \int\left(\frac{d}{dx}\log x . \int x^2 dx\right)dx$$

$$= \log x \times \frac{x^3}{3} - \int \frac{1}{x}\times\frac{x^3}{3}dx$$

$= \frac{x^3}{3}\log x - \frac{1}{3}\int x^2 dx$

$= \frac{x^3}{3}\log x - \frac{1}{3} \times \frac{x^3}{3} + C$

$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C$

12. Evaluate: $\int_0^3 f(x)\,dx$, where $f(x) = \begin{cases} x+1, & x<1 \\ 2x, & x \geq 1 \end{cases}$

Sol. Given, $f(x) = \begin{cases} x+1, & x < 1 \\ 2x, & x \geqslant 1 \end{cases}$

$$\therefore \int_0^3 f(x)dx = \int_0^1 (x+1)dx + \int_1^3 2x\,dx$$

$$= \left[\frac{x^2}{2} + x\right]_0^1 + \left[2\frac{x^2}{2}\right]_1^3$$

$$= \frac{1}{2} + 1 - 0 + (3)^2 - (1)^2 = \frac{19}{2}$$

13. Evaluate: $\int_{-2}^2 \frac{1}{1+\sqrt{e^x}}\,dx$

Sol. We have,

$$I = \int_{-2}^{2} \frac{1}{1+\sqrt{e^x}} \quad \ldots(i)$$

$$\Rightarrow \quad I = \int_{-2}^{2} \frac{1}{1+\sqrt{e^{-x}}}$$

Using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_{-2}^{2} \frac{\sqrt{e^x}}{\sqrt{e^x}+1}\,dx \quad \ldots(ii)$$

Adding (*i*) and (*ii*), we have

$$2I = \int_{-2}^{2} \frac{1+\sqrt{e^x}}{1+\sqrt{e^x}}\,dx = \int_{-2}^{2} dx = [x]_{-2}^{2}$$

$\Rightarrow 2I = 2 - (-2) = 4$

$\therefore \; I = 2$

14. Evaluate: $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\,dx$

Sol. We have, $I = \int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right)e^{2x}\,dx$

Let $2x = t \quad \Rightarrow \quad 2\,dx = dt$

when $x = 1 \quad \Rightarrow \quad t = 2$

$x = 2 \quad \Rightarrow \quad t = 4$

$$\therefore \quad I = \int_2^4 \left(\frac{2}{t} - \frac{1}{2} \times \frac{4}{t^2}\right)e^t \times \frac{dt}{2}$$

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt$$

$$= \int_2^4 \left[\frac{1}{t} + \left(\frac{-1}{t^2}\right)\right] e^t dt$$

$$= \left[e^t \cdot \frac{1}{t}\right]_2^4 = \frac{e^4}{4} - \frac{e^2}{2} = \frac{e^2(e^2-2)}{4}$$

15. Evaluate: $\int \frac{1-x}{x(1-2x)} dx$

Sol. We have,

$$I = \int \frac{1-x}{x(1-2x)} dx$$

Let $\frac{1-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x}$ ...(*i*)

$\Rightarrow \frac{1-x}{x(1-2x)} = \frac{A - 2Ax + Bx}{x(1-2x)}$

$\Rightarrow 1 - x = A + (-2A + B).x$

On equating we have

$A = 1$ and $-2A + B = -1$

$\Rightarrow (-2 \times 1) + B = -1$

$\Rightarrow B = -1 + 2 = 1$

$\therefore B = 1$

from (*i*), we have

$$\frac{1-x}{x(1-2x)} = \frac{1}{x} + \frac{1}{1-2x}$$

$$\int \frac{1-x}{x(1-2x)} dx = \int \frac{1}{x} dx + \int \frac{1}{1-2x} dx$$

$$= \log|x| + \frac{\log|1-2x|}{-2} + C$$

$$= \log|x| - \frac{1}{2}\log|1-2x| + C$$

16. Evaluate: $\int_0^1 \frac{3t^2}{(1+t^3)(2+t^3)} dt$

Sol. We have,

$$I = \int_0^1 \frac{3t^2}{(1+t^3)(2+t^3)} dt$$

Let $t^3 = z \Rightarrow 3t^2 dt = dz$

When $t = 0 \Rightarrow z = 0$

$t = 1 \Rightarrow z = 1$

$$\therefore I = \int_0^1 \frac{dz}{(1+z)(2+z)} = \int_0^1 \frac{(2+z)-(1+z)}{(1+z)(2+z)} dz$$

$$= \int_0^1 \frac{1}{1+z} dz - \int_0^1 \frac{1}{2+z} dz$$

$$= \left[\log|1+z| - \log|2+z|\right]_0^1$$

$$= \left[\log\left|\frac{1+z}{2+z}\right|\right]_0^1 = \log\frac{2}{3} - \log\frac{1}{2} = \log\frac{2}{3}\times 2 = \log\frac{4}{3}$$

17. Evaluate: $\int_0^4 (|x|+|x-2|+|x-4|)\, dx$

Sol. Let $I = \int_0^4 (|x|+|x-2|+|x-4|)\, dx$

$$= \int_0^4 \left(|x|\, dx + \int_0^4 |x-2|\, dx + \int_0^4 |x-4|\, dx\right.$$

$$= \int_0^4 \left(|x|\, dx + \left[\int_0^2 |x-2|\, dx + \int_2^4 |x-2|\, dx\right] + \int_0^4 |x-4|\, dx\right. \quad \text{[By properties]}$$

$$= \int_0^4 x\, dx + \int_0^2 -(x-2))dx + \int_2^4 (x-2))dx + \int_0^4 -(x-4)dx$$

$$\left[\begin{array}{l} \because |x| = x, \quad \text{if } 0 \le x \le 4 \\ \quad |x-2| = -(x-2), \quad \text{if } 0 \le x \le 2 \\ \quad |x-2| = (x-2), \quad \text{if } 2 \le x \le 4 \\ \quad |x-4| = -(x-4), \quad \text{if } 0 \le x \le 4 \end{array}\right]$$

$$= \left[\frac{x^2}{2}\right]_0^4 - \left[\frac{(x-2)^2}{2}\right]_0^2 + \left[\frac{(x-2)^2}{2}\right]_2^4 - \left[\frac{(x-4)^2}{2}\right]_0^4$$

$$= \frac{1}{2}\times 16 - \frac{1}{2}\times(0-4) + \frac{1}{2}(4-0) - \frac{1}{2}\times(0-16) = 8+2+2+8 = 20$$

18. Evaluate: $\int \frac{1}{\sqrt{3x^2+2x-1}}\, dx$

Sol. We have,

$$I = \int \frac{1}{\sqrt{3x^2+2x-1}} dx$$

$$I = \int \frac{1}{\sqrt{3\left(x^2+\frac{2}{3}x-\frac{1}{3}\right)}}$$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2+\frac{2}{3}x+\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2-\frac{1}{3}}}\, dx$$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{\left(x+\frac{1}{3}\right)^2-\frac{4}{9}}}\, dx$$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{\left(x+\frac{1}{3}\right)^2-\left(\frac{2}{3}\right)^2}}\, dx$$

$$= \frac{1}{\sqrt{3}}\log\left|\left(x+\frac{1}{3}\right)+\sqrt{\left(x+\frac{1}{3}\right)^2-\left(\frac{2}{3}\right)^2}\right| + C$$

$$= \frac{1}{\sqrt{3}}\log\left|\left(x+\frac{1}{3}\right)+\sqrt{x^2+\frac{2x}{3}-\frac{1}{3}}\right| + C$$

19. Evaluate: $\int \frac{5x+4}{(x^2-1)(x+2)}dx$

Sol. We have,

$$I = \int \frac{5x+4}{(x^2-1)(x+2)}dx$$

$$I = \int \frac{5x+4}{(x-1)(x+1)(x+2)}dx$$

Let $\frac{5x+4}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$

Where, $A = \left.\frac{5x+4}{(x+1)(x+2)}\right|_{x=1} \Rightarrow A = \frac{9}{2\times 3} = \frac{3}{2}$

$$B = \left.\frac{5x+4}{(x-1)(x+2)}\right|_{x=-1} \Rightarrow B = \frac{-1}{-2\times 1} = \frac{1}{2}$$

and $C = \left.\frac{5x+4}{(x-1)(x+1)}\right|_{x=-2} \Rightarrow C = \frac{-6}{-3\times -1} = -2$

$$\therefore \int \frac{5x+4}{(x-1)(x+1)(x+2)}dx = \frac{3}{2}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x+1}dx - 2\int \frac{1}{x+2}dx$$

$$= \frac{3}{2}\log|x-1| + \frac{1}{2}\log|x+1| - 2\log|x+2| + C$$

20. Evaluate: $\int_1^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{6-x}}dx$

Sol. We have,

$$I = \int_1^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{6-x}}dx \quad ...(i)$$

Using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_1^5 \frac{\sqrt{6-x}}{\sqrt{6-x}+\sqrt{x}}dx \quad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_1^5 \frac{\sqrt{x}+\sqrt{6-x}}{\sqrt{x}+\sqrt{6-x}}dx = \int_1^5 dx$$

$$\Rightarrow \quad 2I = [x]_1^5 = 5-1 = 4$$

$$\therefore \quad I = \frac{4}{2} = 2$$

21. The demand function for a commodity is $p = 80 - 3x - x^2$. Find the consumers surplus for $p = 40$.

Sol. Given that demand function is $p = 80 - 3x - x^2$

When $p_0 = 40$ we have

$$40 = 80 - 3x - x^2 \Rightarrow x^2 + 3x - 40 = 0$$

$$x^2 + 8x - 5x - 40 = 0 \Rightarrow x(x+8) - 5(x+8) = 0$$

$$(x+8)(x-5) = 0$$

$\Rightarrow \quad x - 5 = 0 \quad (x + 8 \neq 0) \Rightarrow x \neq -8$

$\therefore \quad x = 5 \quad \Rightarrow \quad x_0 = 5$

$\therefore \quad p_0 x_0 = 40 \times 5 = 200$

Consumers' surplus

$$\text{C.S} = \int_0^5 (80 - 3x - x^2)dx - 200$$

$$= \left[80x - \frac{3x^2}{2} - \frac{x^3}{3}\right]_0^5 - 200$$

$$= 400 - \frac{75}{2} - \frac{125}{3} - 200$$

$$= \frac{725}{6}$$

22. If the supply function is $p = 3x^2 + 10$ and $x_0 = 4$, find the producer's surplus.

Sol. Given supply function, $p = 3x^2 + 10$ and $x_0 = 4$

$\therefore \quad p_0 = 3 \times (4)^2 + 10 = 58$

$\therefore$ Producers' surplus (P. S) $= p_0 x_0 - \int_0^{x_0} (3x^2 + 10)dx = 58 \times 4 - \int_0^4 (3x^2 + 10)dx$

$$= 58 \times 4 - [x^3 + 10x]_0^4$$

$$= 232 - (64 + 40) = 128$$

QUESTIONS FOR PRACTICE SOLUTIONS

Chapter-2 Differential Equations and Modelling

1. $x^5 \dfrac{dy}{dx} = -y^5$

Sol. Given differential equation,

$$x^5 \frac{dy}{dx} = -y^5$$

$$\frac{dy}{y^5} = -\frac{dx}{x^5}$$

On integrating both sides, we have,

$$\int \frac{dy}{y^5} = -\int \frac{dx}{x^5} \quad \Rightarrow \quad \int y^{-5} dy = -\int x^{-5} dx$$

$$\Rightarrow \quad \frac{y^{-5+1}}{-5+1} = -\left(\frac{x^{-5+1}}{-5+1} + C_1\right)$$

$$\Rightarrow \quad -\frac{1}{4y^4} = \frac{1}{4x^4} - C_1$$

$$\Rightarrow \quad \frac{1}{x^4} + \frac{1}{y^4} = 4C_1$$

$$\Rightarrow \quad \frac{1}{x^4} + \frac{1}{y^4} = C \quad (\text{Where } 4C_1 = C)$$

$$\Rightarrow \quad x^4 + y^4 = C\,x^4 y^4$$

which is the required solution.

2. $\dfrac{dy}{dx} = \dfrac{y}{x}$

Sol. Given differential equation, $\dfrac{dy}{dx} = \dfrac{y}{x}$

$$\frac{dy}{y} = \frac{dx}{x}$$

On integrating, we have,

$$\Rightarrow \quad \int \frac{dy}{y} = \int \frac{dx}{x} \quad \Rightarrow \quad \log y = \log x + \log C$$

$$\Rightarrow \quad \log y = \log Cx$$

$$\Rightarrow \quad y = Cx$$

Which is the required solution.

3. $\dfrac{dy}{dx} = 2^{y-x}$

Sol. Given differential equation,

$$\frac{dy}{dx} = 2^{y-x}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2^y \times 2^{-x} = \frac{2^y}{2^x} \quad \Rightarrow \quad \int \frac{dy}{2^y} = \int \frac{dx}{2^x} \quad \Rightarrow \quad \int 2^{-y}\, dy = \int 2^{-x} dx$$

$$\Rightarrow \quad \frac{2^{-y}}{\log 2} = \frac{2^{-x}}{\log 2} + C$$

$$\Rightarrow \quad 2^{-y} = 2^{-x} + C \log 2 \qquad \Rightarrow \quad 2^{-x} - 2^{-y} = -C \log 2$$

$$\Rightarrow \quad 2^{-x} - 2^{-y} = k, \qquad \text{(Where } k = -C \log 2\text{)}$$

Which is the required solution.

4. $\frac{dy}{dx} = (e^x + 1)y$

Sol. Given differential equation,

$$\frac{dy}{dx} = (e^x + 1)y$$

$$\frac{dy}{y} = (e^x + 1)dx$$

On integrating both sides

$$\int \frac{dy}{y} = \int (e^x + 1)dx$$

$\Rightarrow \quad \log|y| = e^x + x + C$ which is the required solution.

5. $\frac{dy}{dx} = \frac{x+1}{2-y}$

Sol. Given differential equation,

$$\frac{dy}{dx} = \frac{x+1}{2-y}$$

$$\Rightarrow (2-y)dy = (x+1)dx$$

On integrating both sides, we have

$$\Rightarrow \quad \int (2-y)dy = \int (x+1)dx$$

$$\Rightarrow \quad 2y - \frac{y^2}{2} = \frac{x^2}{2} + x + C_1 \qquad \Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} + x - 2y + C_1 = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2x - 4y + 2C_1 = 0 \qquad \Rightarrow \quad x^2 + y^2 + 2x - 4y + C = 0 \qquad \text{(where } C = 2C_1\text{)}$$

which is the required solution.

6. Find the general solution of the differential equation:

$$e^x \sqrt{1-y^2}\, dx + \frac{y}{x} dy = 0$$

Sol. Given differential equation,

$$e^x \sqrt{1-y^2}\, dx + \frac{y}{x} dy = 0$$

$$\Rightarrow \quad e^x \sqrt{1-y^2}\, dx = -\frac{y}{x} dy \qquad \Rightarrow \quad x\, e^x dx = -\frac{y}{\sqrt{1-y^2}} dy$$

On integrating both sides, we have

$$\Rightarrow \quad \int x\, e^x dx = \int \frac{-y}{\sqrt{1-y^2}} dy \qquad \Rightarrow \quad x \int e^x dx - \int \left(\frac{dx}{dx} . \int e^x dx\right) dx = \frac{1}{2} \int \frac{-2y}{\sqrt{1-y^2}} dy$$

$$\Rightarrow \quad x\, e^x - \int e^x dx = \frac{1}{2} \int \frac{dt}{\sqrt{t}}, \text{ where } 1 - y^2 = t$$

$$\Rightarrow \quad x\, e^x - e^x = \sqrt{t} + C \qquad \Rightarrow \quad (x-1)e^x = \sqrt{1-y^2} + C$$

Which is the required solution.

7. Solve: $(x+1)\dfrac{dy}{dx} = 2xy$, given that $y(2) = 3$

Sol. Given differential equation:

$$(x+1)\frac{dy}{dx} = 2xy;\ y(2) = 3$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{2x}{x+1}dx$$

On integrating both sides, we have

$$\Rightarrow \quad \int \frac{dy}{y} = 2\int \frac{x}{x+1}dx \qquad \Rightarrow \quad \int \frac{dy}{y} = 2\int \frac{(x+1)-1}{(x+1)}dx$$

$$\Rightarrow \quad \log|y| = 2\int dx - 2\int \frac{1}{x+1}dx$$

$$\Rightarrow \quad \log|y| = 2x - 2\log|x+1| + \log C \qquad \ldots(i)$$

It is given that $y(2) = 3$

$\log 3 = 2 \times 2 - 2\log 3 + \log C$

$3\log 3 - 4 = \log C$

Putting the value of log C in (*i*), we get

$\log|y| = 2x - 2\log|x+1| + 3\log 3 - 4$

Which is the required solution.

8. Solve: $y \log y\, dx - x\, dy = 0$

Sol. Given differential equation,

$$y \log y\, dx - x\, dy = 0$$

$$\Rightarrow \quad y \log y\, dx = x\, dy$$

$$\Rightarrow \quad \int \frac{dx}{x} = \int \frac{dy}{y\log y}$$

$$\Rightarrow \quad \log|x| = \log|\log y| + \log C_1$$

$$\Rightarrow \quad \log|x| = \log|C_1 \log y|$$

$$\Rightarrow \quad x = C_1 \log y \quad \Rightarrow \quad \log y = \frac{1}{C_1}x = Cx, \quad \text{Where } \frac{1}{C_1} = C$$

$$\Rightarrow \quad y = e^{Cx}$$

9. Find a particular solution of the differential equation: $(x^3 + x^2 + x + 1)\dfrac{dy}{dx} = 2x^2 + x$; $y = 1$ where $x = 0$.

Sol. Given differential equation,

$$(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$$

$$dy = \frac{2x^2 + x}{x^3 + x^2 + x + 1}dx$$

On integrating both sides, we have

$$\int dy = \int \frac{2x^2 + x}{x^3 + x^2 + x + 1}dx$$

$$\Rightarrow \quad y = \int \frac{2x^2 + x}{x^2(x+1) + 1(x+1)}dx \qquad \Rightarrow \quad y = \int \frac{2x^2 + x}{(x+1)(x^2+1)}dx$$

Let $\dfrac{2x^2+x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

$\Rightarrow \quad 2x^2 + x = Ax^2 + A + Bx^2 + Bx + Cx + C$

$\Rightarrow \quad 2x^2 + x = (A+B)x^2 + (B+C)x + (A+C)$

On equating $\quad A + B = 2$

$B + C = 1$

$A + C = 0$

On solving these, we get

$$A = \frac{1}{2}, B = \frac{3}{2} \text{ and } C = -\frac{1}{2}$$

$\therefore \quad y = \dfrac{1}{2}\displaystyle\int \frac{1}{x+1}dx + \int \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1}\, dx$

$\Rightarrow \quad y = \dfrac{1}{2}\log|x+1| + \dfrac{3}{2\times 2}\displaystyle\int \frac{2x}{x^2+1}dx - \frac{1}{2}\int \frac{1}{x^2+1}dx$

$\Rightarrow \quad y = \dfrac{1}{2}\log\left|x+1\right| + \dfrac{3}{4}\log\left|x^2+1\right| - \dfrac{1}{2}\tan^{-1}x + C$

When $\quad x = 0, y = 1$

$\therefore \quad 1 = \dfrac{1}{2}\log 1 + \dfrac{3}{4}\log 1 - \dfrac{1}{2}\tan^{-1} 0 + C$

$\Rightarrow \quad 1 = 0 + 0 - 0 + \text{C}$

$\Rightarrow \quad C = 1$

$\therefore \quad y = \dfrac{1}{2}\log|x+1| + \dfrac{3}{4}\log|x^2+1| - \dfrac{1}{2}\tan^{-1}x + 1$

$y = \dfrac{2}{4}\log|x+1| + \dfrac{3}{4}\log(x^2+1) - \dfrac{1}{2}\tan^{-1}x + 1$

$y = \dfrac{1}{4}\log(x+1)^2 + \dfrac{1}{4}\log(x+1)^3 - \dfrac{1}{2}\tan^{-1}x + 1$

$y = \dfrac{1}{4}[\log(x+1)^2 + \log(x+1)^3] - \dfrac{1}{2}\tan^{-1}x + 1$

$y = \dfrac{1}{4}\log[(x+1)^2(x+1)^3] - \dfrac{1}{2}\tan^{-1}x + 1$

10. For the differential equation $xy\dfrac{dy}{dx} = (x+2)(y+2)$ find the solution curve passing through the point (1, –1).

Sol. Given differential equation,

$$xy\frac{dy}{dx} = (x+2)(y+2)$$

$\Rightarrow \quad \dfrac{ydy}{y+2} = \dfrac{x+2}{x}dx$

On integrating both sides

$\Rightarrow \quad \displaystyle\int \frac{y\,dy}{y+2} = \int \frac{x+2}{x}dx$

$\Rightarrow \quad \displaystyle\int \frac{(y+2-2)}{(y+2)} = \int\left(1 + \frac{2}{x}\right)dx$

$\Rightarrow \quad \displaystyle\int\left(1 - \frac{2}{y+2}\right)dy = x + 2\log x + C$

$\Rightarrow \quad y - 2\log|y + 2| = x + 2\log x + C \quad \ldots(i)$

It is given that $x = 1, y = -1$

$\therefore \quad -1 - 2\log 1 = 1 + 2\log 1 + C$

$\Rightarrow \quad -1 - 2 \times 0 = 1 + 0 + C$

$C = -2$

From (i), we get

$y - 2\log|y + 2| = x + 2\log x - 2$

$\Rightarrow \quad y - x + 2 = 2\log x + 2\log|y + 2|$

$\Rightarrow \quad y - x + 2 = \log x^2 (y + 2)^2$

$\therefore \quad y - x + 2 = \log x^2 (y + 2)^2$

11. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (–4, –3). Find the equation of the curve given that it passes through (–2, 1).

Sol. We have,

$$\frac{dy}{dx} = 2\left(\frac{y+3}{x+4}\right)$$

$$\Rightarrow \quad \frac{dy}{y+3} = 2\frac{dx}{x+4}$$

On integrating both sides, we have

$$\Rightarrow \quad \int \frac{dy}{y+3} = 2\int \frac{dx}{x+4}$$

$$\therefore \quad \log|y + 3| = 2\log|x + 4| + C \quad \ldots(i)$$

Since the curve passes through (–2, 1)

$$\therefore \quad \log|1 + 3| = 2\log|-2 + 4| + C$$

$$\Rightarrow \quad 2\log 2 = 2\log 2 + C$$

$$\Rightarrow \quad C = 0$$

From (i), we have

$$\log|y + 3| = \log|x + 4|^2$$

$$\Rightarrow \quad (y + 3) = (x + 4)^2$$

12. In a bank, principal increases continuously at the rate of r% per year. Find the value of r if ₹ 100 double itself in 10 years ($\log_e 2 = 0.6931$).

Sol. Let principal be p

$$\therefore \quad \frac{dp}{dt} = \frac{r}{100} \times p$$

$$\Rightarrow \quad \frac{dp}{p} = \frac{r}{100}\,dt$$

On integrating both sides, we have

$$\Rightarrow \quad \int \frac{dp}{p} = \frac{r}{100}\int dt$$

$$\Rightarrow \quad \log p = \frac{r}{100}.t + C \quad \ldots(i)$$

When $t = 0, p = 100$

$$\log 100 = 0 + C$$

$\therefore \quad C = \log 100$

From (*i*), we get

$$\log p = \frac{r}{100}t + \log 100$$

$\Rightarrow \quad \log p - \log 100 = \frac{r}{100}t$

$\Rightarrow \quad \log \frac{p}{100} = \frac{r}{100}t$

$\Rightarrow \quad \frac{p}{100} = e^{\frac{r}{100}}$

Putting $t = 10, p = 200$, we get

$$\frac{200}{100} = e^{\frac{r}{100} \times 10}$$

$\Rightarrow \quad 2 = e^{\frac{r}{10}}$

$\Rightarrow \quad \frac{r}{10} = \log 2$

$\Rightarrow \quad r = 10 \times 0.6931$

$\Rightarrow \quad r = 6.931$

$\Rightarrow \quad r = 6.93\%$

QUESTIONS FOR PRACTICE SOLUTIONS

Chapter-3 Inferential Statistics

1. Find the critical t value for $\alpha = 0.01$ with DF = 22 for a left-tailed test.

Sol. We have,

$\alpha = 0.01$ with degree of freedom = 22

From the t – table, we have

Critical value $t_{22}(0.01) = -2.508$ (left-tailed test)

2. Find the critical t values for $\alpha = 0.10$ with DF = 18 for a two-tailed t test.

Sol. We have,

$\alpha = 0.10$ with degree of freedom = 18

From the t-table (Two tailed test)

Critical values $t_{18}(0.10) = 1.734$

$\therefore$ Critical value are $+1.734$ and -1.734

3. Suppose that a 95% confidence interval states that population mean is greater than 100 and less than 300. How would you interpret this statement? ***[CBSE Question Bank]***

Sol. This means that we are 95% sure that population mean is between 95 and 315.

4. A shoe maker company produces a specific model of shoes having 15 months average lifetime. One of the employees in their R & D division claims to have developed a product that lasts longer. This latest product was worn by 30 people and lasted on average for 17 months. The variability of the original shoe is estimated based on the standard deviation of the new group which is 5.5 months. Is the designer's claim of a better shoe supported by the findings of the trial? Make your decision using two tailed testing using a level of significance of $p < 0.05$

Sol. We have,

$\mu = 15$, $\overline{X} = 17$, $n = 30$ and $s = 5.5$

Null hypothesis H_0: The company produces a specific model of shoes having 15 months average lifetime *i.e.*, difference between the sample mean $\overline{X}$ and the population mean μ is not significant.

Alternate hypothesis H_1: Specific model of shoes having more than 15 months average life time.

$$\therefore \quad t = \frac{\overline{X} - \mu}{s/\sqrt{n}} = \frac{17-15}{\frac{5.5}{\sqrt{30}}} = \frac{2}{5.5} \times \sqrt{30} = 2$$

Calculated $|t|$ value = 2

and from table, $t_{29}(0.05) = 2.045$

Hence, $2 < 2.045$

Calculated t value $< t_{29}(0.05)$

$\therefore$ Null hypothesis is accepted.

6. A fertilizer company packs the bags labelled 50 kg and claims that the mean mass of bags is 50 kg with a standard deviation 1kg. An inspector points out doubt on its weight and tests 60 bags. As a result, he finds that mean mass is 49.6 kg. Is the inspector right in his suspicions?

Sol. We have,

$\mu = 50$, s = 1, $n = 60$ and $\overline{X} = 49.6$

We define,

Null hypothesis H_0: There is no significant difference in mean mass of bags before and after doubts.

Alternate hypothesis H_1: There is significant difference in the mean mass of bags before and after doubts.

Now, $$t = \frac{\overline{X} - \mu}{S/\sqrt{n}} = \frac{49.6 - 50}{\frac{1}{\sqrt{60}}} = -0.4 \times \sqrt{60} = -3.09$$

$\Rightarrow$ $|t| = 3.09$

and tabulated value of t at 5% level of significance

$t_{59}(0.05) = 2$

Calculate $|t| >$ tabulated t

So, we accept alternate hypothesis, *i.e.*, null hypothesis is rejected.

Hence, the inspection is right is his suspicions.

7. The average heart rate for Indians is 72 beats/minute. To lower their heart rate, a group of 25 people participated in an aerobics exercise programme. The group was tested after six months to see if the group had significantly slowed their heart rate. The average heart rate for the group was 69 beats/ minute with a standard deviation of 6.5. Was the aerobics program effective in lowering heart rate? **[*CBSE Question Bank*]**

Sol. We have,

$\mu = 72, n = 25, \overline{X} = 69$ and $s = 6.5$

We define,

Null hypothesis H_0: There is no significant difference in average heart rate before and after aerobics programme.

Alternate hypothesis H_1: There is significant difference in average heart rate before and after aerobics programme.

Now, $$t = \frac{\overline{X} - \mu}{\frac{S}{\sqrt{n}}} = \frac{69 - 72}{\frac{6.5}{\sqrt{25}}} = \frac{-3}{1.3} = \frac{-30}{13} = -2.30$$

$\therefore$ $|t| = 2.30$

$\therefore$ Calculated value of $|t| = 2.30$ and tabulated value of t at 5% level of significance

$\therefore$ $t_{24}(0.05) = 2.064$

We observe that

Calculated value of $|t| >$ tabulated value of t.

$\therefore$ Alternate hypothesis is accepted *i.e.*, null hypothesis is rejected.

Hence, aerobics programme is effective in lowering heart rate.

QUESTIONS FOR PRACTICE SOLUTIONS

Chapter-4 Index Numbers and Time Based Data

1. Construct 3-yearly moving averages from the following data:

Year	2010	2011	2012	2013	2014	2015	2016
Imported cotton consumption in India (in '000 bales)	129	131	106	91	95	84	93

Sol. We have,

Year	**Imported cotton consumption in India (in '000 bales)**	**3-yearly moving totals**	**3 yearly moving averages**
2010	129	–	–
2011	131	366	122.00
2012	106	328	109.33
2013	91	292	97.33
2014	95	270	90.00
2015	84	272	90.66
2016	93	–	–

2. In an influenza epidemic the number of cases diagnosed were:

Date (March)	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Numbers	2	0	5	12	20	27	46	30	31	18	11	5	0	1

Calculate 3-days moving averages.

Sol. We have,

Date (March)	**Numbers**	**3-days moving totals**	**3 days moving averages**
1	2	–	–
2	0	7	2.3
3	5	17	5.6
4	12	37	12.3
5	20	59	19.6
6	27	93	31.0
7	46	103	34.3
8	30	107	35.6
9	31	79	26.3
10	18	60	20.0
11	11	34	11.3
12	5	16	5.3
13	0	6	2.0
14	1	–	–

3. From the following data construct 4-yearly moving averages and determine the trend values. Also, find the short-term fluctuations.

Year	Value
2006	50
2007	36.5
2008	43.0
2009	44.5
2010	38.9
2011	38.1
2012	32.6
2013	41.7
2014	41.1
2015	33.8

Sol. We have,

Year	Value	4-yearly Moving Totals	4-yearly Moving Averages	Short-term Fluctuations $Y - Y_c$
2006	50	–	–	–
2007	36.5	–	–	–
2008	43.0	174.0	43.5	
			42.1125	0.8875
2009	44.5	162.9	40.725	
			40.925	3.575
2010	38.9	164.5	41.125	
			39.825	– 0.925
2011	38.1	154.1	38.525	
			38.175	– 0.075
2012	32.6	151.3	37.825	
			38.1	– 5.5
2013	41.7	153.5	38.375	
			37.8375	3.8625
2014	41.1	149.2	37.3	
2015	33.8	–		

4. Calculate the 5-yearly moving averages of the following time series of steel production:

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
Production (in Tonnes)	351	366	361	362	400	419	410	420	450	500

Sol. We have,

Year	Production (in Tonnes)	5-yearly Moving Totals	5-yearly Moving Averages
2006	351	–	–
2007	366	–	–
2008	361	1840	368
2009	362	1908	381.6
2010	400	1952	390.4
2011	419	2011	402.2
2012	410	2099	419.8
2013	420	2199	439.8
2014	450	–	–
2015	500	–	–

5. The annual rainfall (in mm) was recorded for Cherrapunji, Meghalaya:

Year	Rainfall (in mm)
2001	1.2
2002	1.9
2003	2
2004	1.4
2005	2.1
2006	1.3
2007	1.8
2008	1.1
2009	1.3

Determine the trend of rainfall by 3-year moving averages.

Sol. We have,

Year	Rainfall (in mm)	3-yearly Moving Totals	3-yearly Moving Averages
2001	1.2	–	–
2002	1.9	5.1	1.70
2003	2	5.3	1.76
2004	1.4	5.5	1.83
2005	2.1	4.8	1.60
2006	1.3	5.2	1.73
2007	1.8	4.2	1.40
2008	1.1	4.2	1.40
2009	1.3	–	–

6. Compute the seasonal indices by 4-year moving averages from the given data of production of paper (in thousand tons)

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989
Index number	2450	1470	2151	1800	1210	1950	2300	2500	2480	2680

Sol. We have,

Year	Index Number	4-yearly Moving Totals	4-yearly Moving Averages	Centered moving averages
1980	2450	–	–	–
1981	1470	–	–	–
1982	2150	7870	1967.50	
				1812.50
1983	1800	6630	1657.50	
				1717.50
1984	1210	7110	1777.50	
				1796.25
1985	1950	7260	1815.00	
				1902.50
1986	2300	7960	1990.00	
				2148.75
1987	2500	9230	2307.50	
				2398.75
1988	2480	9960	2490.00	
				–
1989	2680	–	–	

7. Given below is the data of workers welfare expenses (in lakh ₹) in steel industries during 2001-2005. Use method of least squares to:

(*i*) tabulate the trend values

(*ii*) find the best fit for a straight-line trend

(*iii*) compute expected sale trend for year 2006

Year	2001	2002	2003	2004	2005
Welfare expenses (in lakh ₹)	160	185	220	300	510

Sol. We have,

Year (x_i)	Welfare expenses (in lakh ₹) Y	$X = xi - A$ $X = x_i - 2003$	X^2	XY	Trend Values $Y_t = a + bx$
2001	160	–2	4	–320	$275 + 81.5 \times (-2) = 112$
2002	185	–1	1	–185	$275 + 81.5 \times (-1) = 193.5$
2003	220	0	0	0	$275 + 81.5 \times 0 = 275$
2004	300	1	1	300	$275 + 81.5 \times 1 = 356.5$
2005	510	2	4	1020	$275 + 81.5 \times 2 = 438$
$n = 5$	$\Sigma Y = 1375$	$\Sigma X = 0$	$\Sigma X^2 = 10$	$\Sigma XY = 815$	

Now, $a = \frac{\Sigma Y}{n} = \frac{1375}{5} = 275$

$$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{815}{10} = 81.5$$

∴ Equation of the straight line trend is given by

$$Y_t = a + bX$$

⇒ $Y_t = 275 + 81.5X$

According to the line trend, the expected sales for the year 2006.

$$Y = 275 + 81.5 \times (2006 - 2003)$$

$$Y = 275 + 81.5 \times 3 = ₹\ 519.5\text{sss Lakh}$$

8. Fit a straight-line trend by method of least squares for the following data and also find the trend value for year 1998:

Year	1992	1993	1994	1995	1996	1997
Production (in tons)	210	225	275	220	240	235

Sol. We have,

Year (x_i)	**Production (in tons)** Y	$X = \frac{x_i - 1994.5}{0.5}$	X^2	XY	**Trend Values** $Y_t = a + bx = 234 + 1.6x$
1992	210	–5	25	–1050	234 + 1.6 × (–5) = 226
1993	225	–3	9	–675	234 + 1.6 × (–3) = 229.2
1994	275	–1	1	–275	234 + 1.6 × (–3) = 232.4
1995	220	1	1	220	234 + 1.6 × 1 = 235.6
1996	240	3	9	720	234 + 1.6 × 3 = 238.8
1997	235	5	25	1175	234 + 1.6 × 5 = 242
$n = 6$	$\Sigma Y = 1405$	$\Sigma X = 0$	$\Sigma X^2 = 70$	$\Sigma XY = 115$	

Now, $a = \frac{\Sigma Y}{n} = \frac{1405}{6} = 234.1 \simeq 234$

$$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{115}{70} = 1.64 = 1.6$$

∴ Equation of the straight line trend is given by

$$Y_t = a + bX$$

⇒ $Y_t = 234 + 1.6X$

According to the line formed, the predicted trend for 1998 is given by

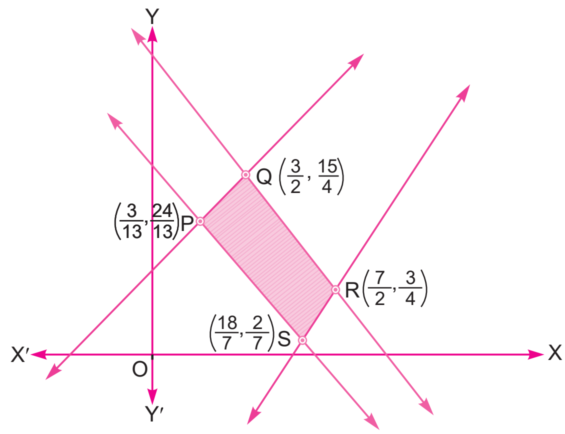
$$Y = 234 + 1.6 \times \left(\frac{1998 - 1994.5}{0.5}\right)$$

$$= 234 + 1.6 \times 7 = 245.2 \text{ tons.}$$

QUESTIONS FOR PRACTICE SOLUTIONS

Chapter-5 Linear Programming

1. In figure, the feasible region (shaded) for a LPP is shown. Determine the maximum and minimum value of $Z = x + 2y$.



Sol. We have corner points of feasible region (shaded) are $P\left(\frac{3}{13}, \frac{24}{13}\right)$, $Q\left(\frac{3}{2}, \frac{15}{4}\right)$, $R\left(\frac{7}{2}, \frac{3}{4}\right)$ and $S\left(\frac{18}{7}, \frac{2}{7}\right)$

Now,

Corner points	$Z = x + 2y$	
$P\left(\frac{3}{13}, \frac{24}{13}\right)$	$\frac{3}{13} + \frac{48}{13} = \frac{51}{13}$	
$Q\left(\frac{3}{2}, \frac{15}{4}\right)$	$\frac{3}{2} + \frac{15}{2} = 9$	← Maximum
$R\left(\frac{7}{2}, \frac{3}{4}\right)$	$\frac{7}{2} + \frac{3}{2} = 5$	
$S\left(\frac{18}{7}, \frac{2}{7}\right)$	$\frac{18}{7} + \frac{4}{7} = \frac{22}{7}$	← Minimum

$\therefore$ Maximum value of $Z = 9$ at $\left(\frac{3}{2}, \frac{15}{4}\right)$

and, minimum value of $Z = \frac{22}{7} = 3\frac{1}{7}$ at $\left(\frac{18}{7}, \frac{2}{7}\right)$

2. Solve the following linear programming problem graphically:

Minimise $\quad Z = x - 5y + 20$

Subject to constraints: $x - y \geq 0, -x + 2y \geq 2;$

$x \geq 3, y \leq 4, x, y \geq 0$

Sol. We have,

Minimise objective function $Z = x - 5y + 20$

Subject to the constraints

$$x - y \geq 0 \quad \ldots(i)$$

$$-x + 2y \geq 2 \quad \ldots(ii)$$

$$x \geq 3 \quad \ldots(iii)$$

$$x \leq 4 \qquad \ldots(iv)$$
$$x, y \geq 0 \qquad \ldots(v)$$

On plotting (*i*), (*ii*), (*iii*), (*iv*) and (*v*), we get the required feasible (shaded) region *ABCD* whose Corner points are

$$A\left(3, \frac{5}{7}\right), B(6, 4), C(4, 4) \text{ and } D(3, 3)$$

Corner points	$Z = x - 5y + 20$
$A\left(3, \frac{5}{2}\right)$	$\frac{21}{2}$
$B(6, 4)$	6
$C(4, 4)$	4 ← Minimum
$D(3, 3)$	8

Minimum value of Z is 4 at $x = 4$ and $y = 4$

3. Solve the following LPP:

Maximise $\quad Z = 5x_1 + 7x_2,$

Subject to constraints: $\quad x_1 + x_2 \leq 4,$

$3x_1 + 8x_2 \leq 24,$

$10x_1 + 7x_2 \leq 35,$

$x_1, x_2 \geq 0.$

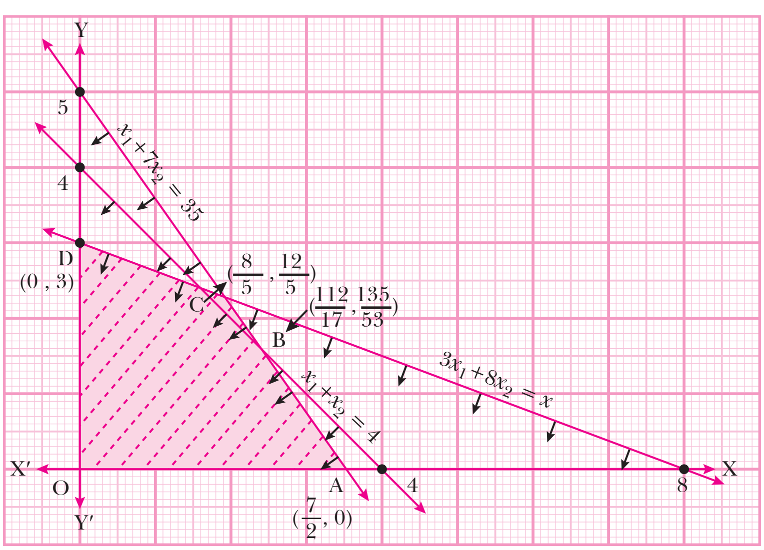
Sol. Given objective function,

Maximise $\quad Z = 5x_1 + 7x_2$

Subject to the constraints

$$x_1 + x_2 \leq 4$$
$$3x_1 + 8x_2 \leq 24$$
$$10x_1 + 7x_2 \leq 35$$
$$x_1, x_2 \geq 0$$

On plotting the given in equations, we got the feasible region (shaded) *OABCD*.



We have corner points

$$O(0, 0),\ A\left(\frac{7}{2}, 0\right), B\left(\frac{112}{17}, \frac{135}{53}\right),\ C\left(\frac{8}{5}, \frac{12}{5}\right) \text{ and } D(0, 3)$$

Now,

Corner points	$Z = 5x_1 + 7x_2$	
$O(0, 0)$	0	
$A\left(\frac{7}{2}, 0\right)$	$\frac{35}{2}$	
$B\left(\frac{7}{3}, \frac{5}{3}\right)$	$\frac{70}{3}$	
$C\left(\frac{8}{5}, \frac{12}{5}\right)$	$\frac{124}{5}$	← Maximum
$D(0, 3)$	21	

$\therefore$ Maximum value of Z is $\frac{124}{5}$ at $\left(\frac{8}{5}, \frac{12}{5}\right)$

4. A manufacturing company makes two models A and B of a product. Each piece of model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing the maximum labour hours available are 180 and 30 respectively. The company makes a profit of ₹8,000 on each piece of model A and ₹12,000 on each piece of model B. To realise a maximum profit formulate above problem in LPP.

Sol. Let x be the number of model A and y be the number of model B.

Thus, we have

	$A(x)$	$B(y)$	
Fabricating	9	12	180
Finishing	1	3	30

$\therefore$ Objective function is to maximise

$$Z = 8000x + 12000y$$

Subject to the constraints

$$9x + 12y \leq 180$$
$$x + 3y \leq 30$$
$$x, y \geq 0$$

5. One kind of cake requires 200 g of flour and 25 g of fat, and another kind of cake requires 100 g of flour and 50 g of fat. To get the maximum number of cakes can be made from 5 kg of flour and 1 kg of fat, formulate the problem in LPP.

Sol. Let x and y be the number of cakes of first and second kind respectively.

$\therefore$ Objective function is $Z_{max} = x + y$ (Z is the total number of GDP)

Subject to the constraints

$$200x + 100y \leq 5000 \Rightarrow 2x + y \leq 50$$
$$25x + 50y \leq 1000 \Rightarrow x + 2y \leq 40$$
$$x, y \geqslant 0$$

6. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹17.50 per package on nuts and ₹7.00 per package on bolts. If he operates his machines for at most 12 hours a day, the formulate the problems in LPP to maximise his profit.

Sol. Let x and y be the number of packages of nuts and bolts produced.

We have

	Nuts	**Bolts**
Machine A	1	3
Machine B	3	1

$\therefore$ Objective function is to maximize profit

$$Z = 17.50x + 7y$$

Subject to the constraints

$$x + 3y \leq 12$$
$$3x + y \leq 12$$
$$x, y \geq 0$$

7. A merchant plans to sell two types of personal computers a desktop model and a portable model that will cost ₹25,000 and ₹40,000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. If he does not want to invest more than ₹70 lakhs and if his profit on the desktop model is ₹4500 and on portable mode is ₹5000, then formulate the problems as LPP to get maximum profit.

Sol. Let number of desktop model be x and number of portable model be y

$\therefore$ Objective function is to maximise profit

$$Z = 4500\,x + 5000\,y$$

Subject to the constraints

$$x + y \leq 250$$
$$25000x + 40000y \leq 7000000 \quad \Rightarrow \quad 5x + 8y \leqslant 1400$$
$$x, y \geq 0$$

8. A man rides his motorcycle at the speed of 50 km/hour. He has to spend ₹2 per km on petrol. If he rides it at a faster speed of 80 km/hour, the petrol cost increases to ₹3 per km. He has at the most ₹120 to spend on petrol and one hour time. He wishes to find the maximum distance that he can travel. Express this problem as a linear programming problem.

Sol. Let a man travels x km at a speed of 50 km/h and y km at a speed of 80 km/h

Let z be the total distance travel (which is maximum)

$\therefore \qquad Z_{max} = x + y$

Now, he has ₹120 to spend on petrol

$\therefore \qquad 2x + 3y \leq 120$

Also, we rides for one hour (at most)

$\therefore \qquad \dfrac{x}{50} + \dfrac{y}{80} \leq 1$

$\Rightarrow \qquad 8x + 5y \leq 400$

Hence, objective function is

Maximum distance, $Z = x + y$

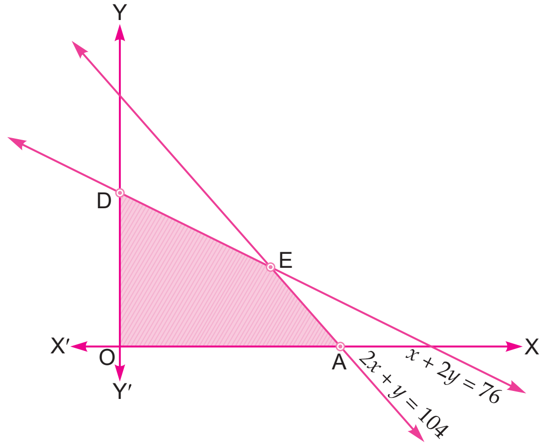
Subject to the constraints

$$2x + 3y \leq 120$$

$8x + 5y \leq 400$

$x, y \geq 0$

9. Determine the maximum value of $Z = 3x + 4y$ of the feasible region (shaded) for a LPP is shown in figure.

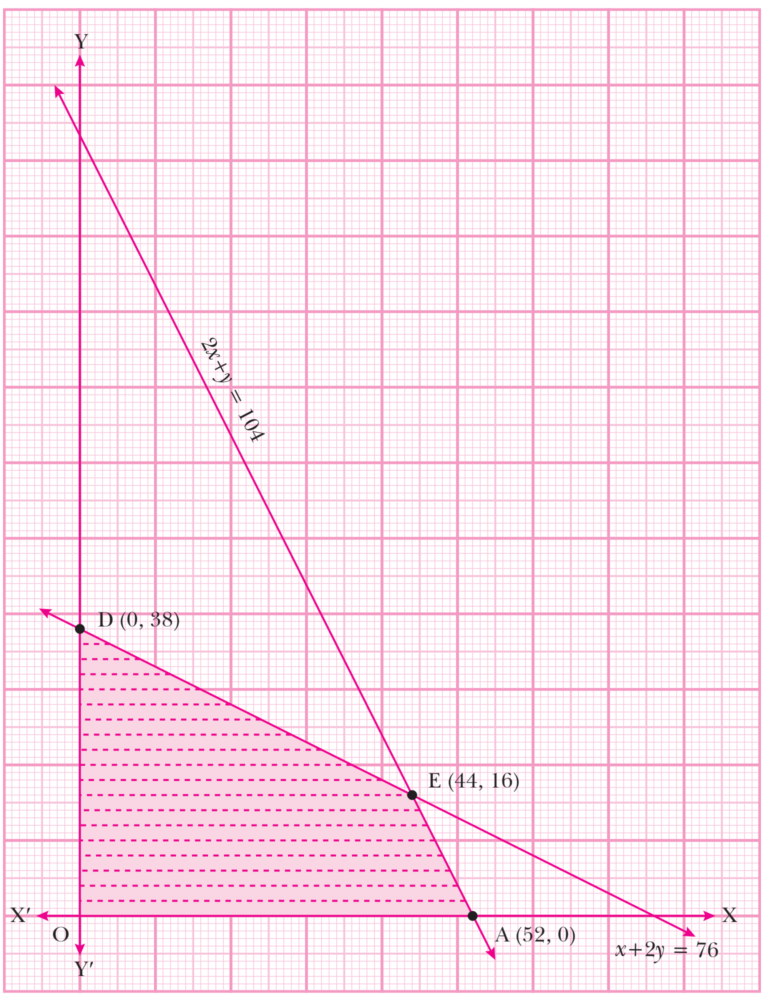


Sol. Point of intersection of

Lines $2x + y = 104$ ...(*i*)

and $x + 2y = 76$ ...(*ii*)

is given by $E(44, 16)$



∴ Corner points of this shaded (feasible) region are 0(0, 0),

$A(52, 0)$, $E(44, 16)$ and $D(0, 38)$

Corner points	$Z = 3x + 4y$
$O(0, 0)$	0
$A(52, 0)$	156
$E(44, 16)$	196 ← Maximum
$D(0, 38)$	152

∴ $Z_{max} = 196$ at $(44, 16)$

10. (Diet problem) *A* dietician has to develop a special diet using two foods *P* and *Q*. Each packet (containing 30 g) of food *P* contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin *A*. Each packet of the same quantity of food *Q* contain 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin *A*. The diet requires at least 240 units of calcium, at least 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin *A*?

Sol. Let dietician uses x packets of food *P* and y packets of food *Q*.

Let *Z* be the amount of vitamin *A*, which have to minimize.

Here $Z = 6x + 3y$ ...(*i*) is objective function

Subject to constraints:

$12x + 3y \geq 240$

$\Rightarrow \quad 4x + y \geq 80 \quad$ ...(*ii*)

$4x + 20y \geq 460$

$\Rightarrow \quad x + 5y \geq 115 \quad$ ...(*iii*)

$6x + 4y \leq 300$

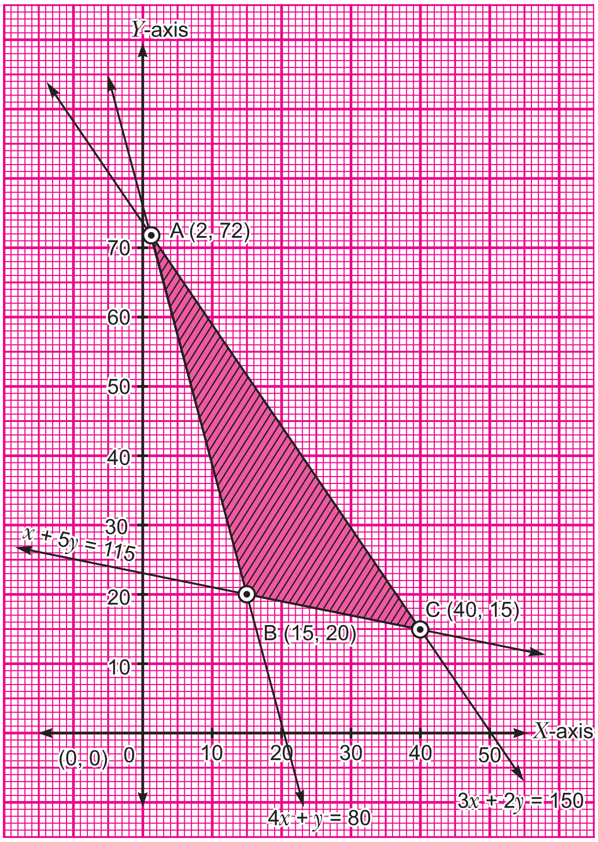
$\Rightarrow \quad 3x + 2y \leq 150 \quad$ ...(*iv*)

$x \geq 0, y \geq 0 \quad$ ...(*ii*)

On plotting graph of above constraints (or inequalities) (*ii*), (*iii*), (*iv*) and (*v*) we get bounded shaded region as feasible region having corner points *A*, *B* and *C*. The coordinates of the corner-points of the feasible region *OABC* are *A*(2, 72), *B*(15, 20) and C(40, 15). These points are obtained by solving the corresponding intersecting lines.

Now, the value of *Z* is evaluated at corner points as

Corner points	$Z = 6x + 3y$
$A(2, 72)$	228
$B(15, 20)$	150 ← Minimum
$C(40, 15)$	285



From table, we get Z is minimum at (15, 20). Hence dietician should use 15 packets of food P and 20 packets of food Q to minimize the vitamin A. Minimum amount of vitamin A is 150 units.

11. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹17.50 per package on nuts and ₹7 per package on bolts. How many packages of each should be produced each day so as to maximize his profits if he operates his machines for at the most 12 hours a day? Form the linear programming problem and solve it graphically.

Sol. Let x package nuts and y package bolts are produced

Let Z be the profit function, which we have to maximize.

Here $Z = 17.50x + 7y$ ... (*i*) is objective function.

And constraints are

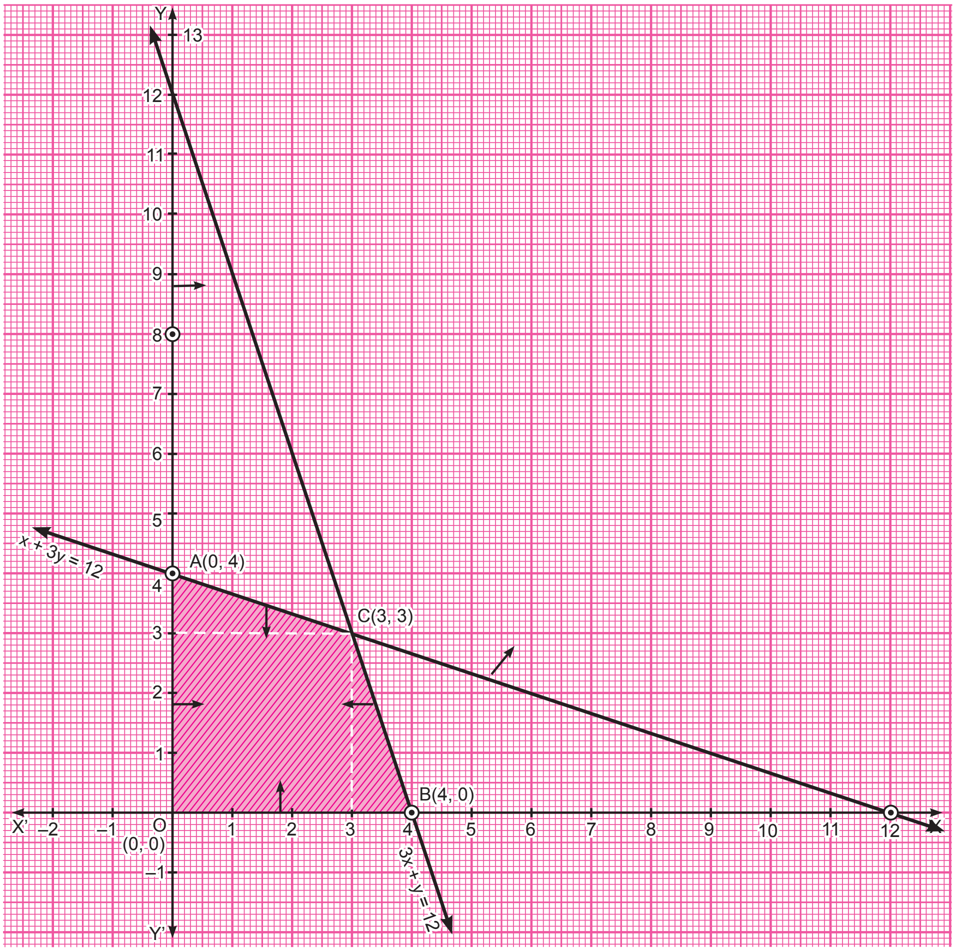
$x + 3y \leq 12 \ldots (ii)$

$3x + y \leq 12 \ldots (iii)$

$x \geq 0 \ldots (iv)$

$y \geq 0 \ldots (v)$

On plotting graph of above constraints or inequalities (*ii*), (*iii*), (*iv*) and (*v*) we get shaded region as feasible region having corner points A, O, B and C.



For coordinate of 'C'

$x + 3y = 12 \ldots (vi)$

$3x + y = 12 \ \ldots (vii)$ are solved

Applying $(vi) \times 3 - (vii)$, we get

$3x + 9y - 3x - y = 36 - 12$

$\Rightarrow 8y = 24 \ \Rightarrow y = 3$ and $x = 3$

Hence coordinate of C are (3, 3).

Now the value of Z is evaluated at corner point as

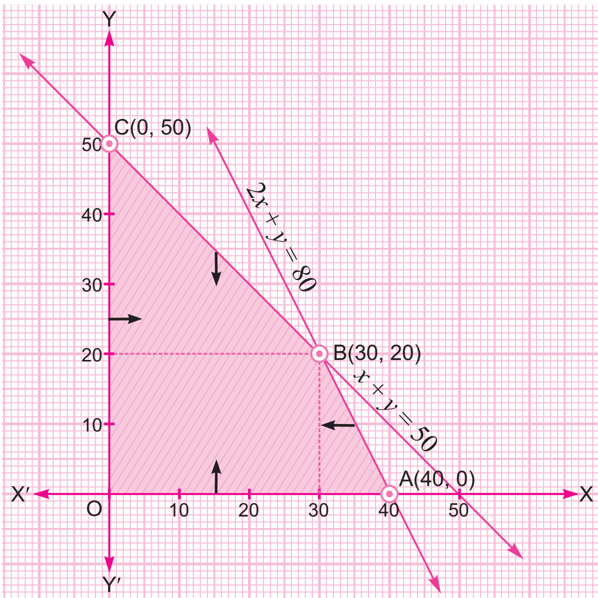
Corner points	$Z = 17.5x + 7y$
(0, 4)	28
(0, 0)	0
(4, 0)	70
(3, 3)	73.5 ← Maximum

Therefore maximum profit is 73.5 when 3 package nuts and 3 package bolt are produced.

12. A cooperative society of farmers has 50 hectares of land to grow two crops A and B. The profits from crops A and B per hectare are estimated as ₹ 10,500 and ₹ 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops A and B at the rate of 20 litres and 10 litres per hectare, respectively. Further not more than 800 litres of herbicide should be used in order to protect fish and wildlife using a pond which collects drainage from this land. Keeping in mind that the protection of fish and other wildlife is more important than earning profit, how much land should be allocated to each crop so as to maximize the total profit? Form an LPP from the above and solve it graphically.

Sol. Let x and y hectares of land be allocated to crop A and B respectively. If Z is the profit then

$$Z = 10500x + 9000y \qquad \ldots(i)$$



We have to maximize Z subject to the constraints:

$$x + y \leq 50 \qquad \ldots(ii)$$

$$20x + 10y \leq 800 \Rightarrow 2x + y \leq 80 \qquad \ldots(iii)$$

$$x \geq 0, y \geq 0 \qquad \ldots(iv)$$

Table for $x + y = 50$

x	0	50
y	50	0

Table for $2x + y = 80$

x	0	40
y	80	0

The graph of system of inequalities (*ii*) to (*iv*) are drawn, which gives feasible region *OABC* with corner points *O* (0, 0), *A* (40, 0), *B* (30, 20) and *C* (0, 50).

Feasible region is bounded.

Now,

Corner points	Z = 10500*x* + 9000*y*	
O (0, 0)	0	
A (40, 0)	420000	
B (30, 20)	495000	← Maximum
C (0, 50)	450000	

Hence, the co-operative society of farmers will get the maximum profit of ₹ 495000 by allocating 30 hectares for crop *A* and 20 hectares for crop *B*.

13. A retired person wants to invest an amount of ₹50,000. His broker recommends investing in two type of bonds '*A*' and '*B*' yielding 10% and 9% return respectively on the invested amount. He decides to invest at least 20,000 in bond '*A*' and at least 10,000 in bond '*B*'. He also wants to invest at least as much in bond '*A*' as in bond '*B*'. Solve this linear programming problem graphically to maximise his returns.

Sol. Let retired person invest ₹x in bond *A* and y in bond *B*. The linear programming problem is to maximise his returns '*Z*'.

Here, $$Z = \frac{10}{100}x + \frac{9x}{100} \quad ...(i)$$

Subject to constraints

$$x + y \leq 50000 \quad ...(ii)$$

$$x \geq 20000 \quad ...(iii)$$

$$y \geq 10000 \quad ...(iv)$$

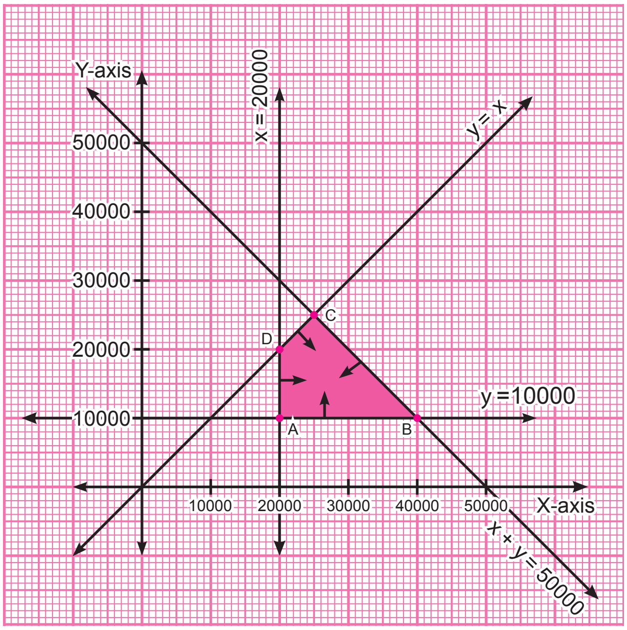
$$y \leq x \quad ...(v)$$

$$x, y \geq 0 \quad ...(vi)$$

To Solve the LPP, the graph is plotted as follows.

The shaded region in the graph is the feasible region with corner points

A (20000, 10000); *B* (40000, 10000); *C* (25000, 25000) and *D* (20000, 20000)



The value of *Z* at corner point is given by

Corner points	$Z = \frac{10x}{100} + \frac{9x}{100}$
$A(20000, 10000)$	$2000 + 900 = 2900$
$B(40000, 10000)$	$4000 + 900 = 4900$ ← Maximum
$C(25000, 25000)$	$2500 + 2250 = 4750$
$D(20000, 20000)$	$2000 + 1800 = 3800$

Hence, for maximum returns of ₹4900, retired person has to invest ₹40000 in bond A and ₹10000 in bond B.

QUESTIONS FOR PRACTICE SOLUTIONS

Chapter-6 Financial Mathematics

1. A company intends to create a sinking fund to replace at the end of 20th year assets costing ₹ 5,00,000. Calculate the amount to be retained out of profits every year if the interest rate is 5% [Given $(1.05)^{20} = 2.6532$].

Sol. Let ₹ R be the amount to be retained out of profits every year for 20 years to accumulate ₹5,00,000.

We have,

$$S = 5,00,000, \quad i = \frac{5}{100} = 0.05 \quad \text{and} \quad n = 20.$$

$$\therefore \quad R = \frac{iS}{(1+i)^n - 1}$$

$$= \frac{0.05 \times 5,00,000}{(1+0.5)^{20} - 1} = \frac{25,000}{(1.05)^{20} - 1}$$

$$= \frac{25,000}{2.6532 - 1} = \frac{25,000}{1.6532} = 15,122.18$$

2. At what rate converted semi-annually will the present value of a perpetuity of ₹450 payable at the end of each 6 months be ₹20,000?

Sol. Let r be the interest rate converted semi-annually. Then i, the interest rate per period is $\frac{r}{2}$

Since, $$P = \frac{R}{i}$$

where $P = 20,000$ and $R = 450$

We have $$i = \frac{R}{P} = \frac{450}{20,000} = 0.0225$$

$$\frac{r}{2} = 0.0225$$

$$r = 0.045 \text{ or } 4.5\ \%$$

3. Mr. X plans to save amount for higher studies of his son, required after 10 years. He expects the cost of these studies to be ₹1,00,000. How much should he save at the beginning of each year to accumulate this amount at the end of 10 years, if the interest rate is 12% compounded annually?

Sol. Let the size of each annual payment be ₹R. These payments represent annuity due consisting 10 annual payments at the rate 0.12 per annum. Thus, using the following formula for the amount of annuity due:

$$A = R\left[S_{\overline{n+1}|i} - 1\right]$$

Where $A = 1,00,000$, $n = 10$ and $i = 0.12$

$$1,00,000 = R\,[S_{110.12} - 1]$$

$$= R\left[\frac{(1.12)^{11} - 1}{0.12}\right]$$

$$= R\,(19.65458)$$

$$R = \frac{1,00,000}{19.65458} = ₹5087.87$$

4. A man buys a house for which he agrees to pay ₹5,000 at the end of each month for 8 years. If money is worth 12% converted monthly, what is the cash price of the house?

[Given $(1.01)^{-96} = 0.3847229701$].

Sol. Let ₹ P be the cash price of the house.

We have,

$$\text{EMI} = ₹\ 5{,}000, \quad n = 12 \times 8 = 96$$

and $$i = \frac{12}{1200} = 0.01$$

∴ $$E = \frac{Pi}{1-(1+i)^{-n}}$$

$$P = \frac{E}{i}\{1-(1+i)^{-n}\}$$

$$P = \frac{5000}{0.01}\{i-(1.01)^{-96}\}$$

$$P = 5{,}00{,}000(1 - 0.3847229701)$$

$$= 5{,}00{,}000 \times 0.6152770299$$

$$P = ₹307{,}638.51$$

Hence, the cash price of the house is ₹ 307638.51.

5. Mr. X takes a loan of ₹2,00,000 with 10% annual interest rate for 5 years. Calculate EMI under Flat Rate System.

Sol. We are given that

$$P = ₹2{,}00{,}000$$

$$I = \frac{10}{100} \times 2{,}00{,}000 \times 5 = ₹\ 1{,}00{,}000$$

$$n = 5 \text{ years} = 5 \times 12 = 60$$

EMI is given by the formula

$$\text{EMI} = \left(\frac{P+I}{n}\right)$$

$$\text{EMI} = \left(\frac{2{,}00{,}000 + 1{,}00{,}000}{60}\right)$$

$$= \frac{3{,}00{,}000}{60} = ₹\ 5000$$

6. What annual rate compounded continuously is equivalent to an effective rate of 10%?

Sol. Let r be the annual rate and r_e be the effective rate.

We know that

$$r = 2.3025 \log(1 + r_e)$$

$$r = 2.3025 \log\left(1 + \frac{10}{100}\right)$$

$$r = 2.3025 \log(1 + 0.1)$$

$$r = 2.3025 \log(1.1)$$

$$r = 2.3025 \times 0.0414 = 0.0953$$

Hence, the annual rate is 9.53%.

7. A bond has face value of ₹10,000 and matures in 15 years at par. The nominal interest is 7%. What is the price of the bond that will yield an effective interest of 8%?
[Given $(1.08)^{-15} = 0.31524170$]

Sol. We have,

Face value (F) = ₹ 10,000

Number of Periods (n) = 15

Annual yield rate (i) = $\frac{8}{100} = 0.08$

Annual dividend = $\frac{7}{100} \times 10,000 =$ ₹ 700

Maturity value (C) = Face value = ₹10,000

∴ Purchase price (V) of the bond is given by

$$V = R\left\{\frac{1-(1+i)^{-n}}{i}\right\} + C(1+i)^{-n}$$

$$V = 700\left\{\frac{1-(1+0.08)^{-15}}{0.08}\right\} + 10000(1+0.08)^{-15}$$

$$V = \frac{70,000}{8}\{1-(1.08)^{-15}\} + 10,000(1.08)^{-15}$$

$$V = \frac{70,000}{8}\{1-0.31524170\} + 10,000 \times 0.31524170$$

$$V = \frac{70,000}{8} \times 0.6847583 + 3152.4170$$

V = ₹ (5991.6351 + 3152.4170)

∴ V = ₹ 9144.05

Hence, purchase price of bond is ₹9144.05.

8. Find the effective rate of interest equivalent to a nominal rate of 6% compounded

(*i*) Semi-annually

(*ii*) Quarterly

(*iii*) Continuously

Sol. (*i*) When compounded semi-annually

We have $r = 0.06$ and $m = 2$

$$r_{eff} = \left(1+\frac{r}{m}\right)^m - 1$$

$$= \left(1+\frac{0.06}{2}\right)^2 - 1$$

$= 0.0609$ or 6.09 %

(*ii*) When compounded quarterly

We have $r = 0.06$ and $m = 4$

$$r_{eff} = \left(1+\frac{r}{m}\right)^m - 1$$

$$= \left(1+\frac{0.06}{4}\right)^4 - 1$$

$= 0.0613$ or 6.13 %

(*iii*) When compounded continuously

$$r_{eff} = e^r - 1 = e^{0.06} - 1$$
$$= 1.0618 - 1$$
$$= 0.0618 \text{ or } 6.18\%$$